Solutions to Higher-Order Boundary-Layer Equations for Flow over a Semi-Infinite Plate

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Introduction

IN a previous paper,[1] a new set of boundary-layer equations was derived containing higher-order corrections to the usual boundary-layer equations. These new equations are regarded as higher-order asymptotic solutions to the Bhatnagar-Gross-Krook model of the Boltzman equation. The method employed is a procedure of coupling the Chapman-Enskog expansion with Prandtl's boundary-layer analysis. It turns out that the lowest order (i.e., first-order) asymptotic solutions lead to the usual boundary-layer equations.[2] In this paper we shall solve these new equations and find higher-order boundary-layer solutions by considering the problem of the laminar boundary-layer flow over a semi-infinite flat plate.

It is well known that the usual theory of boundary layer is the first approximation of the asymptotic solution of the Navier-Stokes equations for large Reynolds numbers.[3] The first approximation to the problem of semi-infinite plate was given by Blasius,[4] Topfer,[5] and Goldstein.[6] Alden[7] was the first to consider higher-order approximation, although his solution was erroneous. Alden's solution was discussed by Goldstein[8] and Imai.[9] Higher-order boundary-layer solutions derived from Navier-Stokes equations have been discussed in detail in the books of Goldstein[10] and Van Dyke.[11]

Analysis

We consider two-dimensional viscous, incompressible fluid flow with uniform velocity U past a semi-infinite plate situated at the positive x axis. The governing equations in dimensionless variables are (see Ref. 1‡):

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}\frac{\partial^2 u}{\partial y^2}+\frac{1}{R}T\left\{\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}-14\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}-4\left(u\frac{\partial^3 u}{\partial x\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right)+6\frac{\partial^4 u}{\partial y^4}\right\}+0(R^{-3/2}) \qquad (2)$$

$$\frac{\partial P}{\partial y}+\frac{T}{R}\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{\partial^2 v}{\partial y^2}\right]+0(R^{-3/2})=0 \qquad (3)$$

where x and u are made dimensionless by referring to an arbitrary length L and the velocity U, respectively. y and v are measured against $\sqrt{T/L}$ and $\sqrt{T/U}$, respectively. The fluid density is constant throughout the entire flowfield. The dimensionless temperature T is also constant. The Reynolds number R is given by $R=UL/\gamma$ where γ is the kinematic viscosity. We assume that R is large.

First, we can eliminate P in Eqs. (2) and (3) by cross differentiation. Then, we let $u=\Psi_y$ and $v=-\Psi_x$ so that Eq. (1) is automatically satisfied and Eqs. (2) and (3) can be combined into a single equation in Ψ. Furthermore, we assume that Ψ can be expanded in a power series of $R^{-1/2}$:

$$\Psi=\Psi_1+R^{-1/2}\Psi_2+R^{-1}\Psi_3+\ldots$$

The first-order equation is then given by

$$\frac{\partial}{\partial y}(\Psi_{1y}\Psi_{1xy}-\Psi_{1x}\Psi_{1yy}-\Psi_{1yyy})=0$$

which can be integrated to yield

$$\Psi_{1yyy}+\Psi_{1x}\Psi_{1yy}-\Psi_{1y}\Psi_{1xy}=C(x) \qquad (4)$$

where $C(x)$ is proportional to the inviscid surface pressure gradient. For our present case, $C(x)=0$. Equation (4) and the following boundary conditions

$$\Psi_1(x,0)=0,\quad \Psi_{1y}(x,0)=0 \quad \text{for } 0<x<\infty,$$

$$\Psi_{1y}(x,\infty)=1$$

constitute the familiar boundary-layer problem whose solution is

$$\Psi_1(x,y)=\sqrt{2x}f(\eta)$$

where $\eta=y/\sqrt{2x}$ and f is the well-known Blasius function satisfying

$$f'''+ff''=0 \qquad (5a)$$

$$f(0)=f'(0)=0,\ f''(0)=\alpha \qquad (5b)$$

$$f\to\eta-\beta \text{ as } \eta\to\infty$$

with $\alpha=0.4696$ and $\beta=1.21678$.

The second-order equation leads to the problem

$$\frac{\partial}{\partial y}(\Psi_{2y}\Psi_{1xy}+\Psi_{2xy}\Psi_{1y}-\Psi_{2x}\Psi_{1yy}-\Psi_{2yy}\Psi_{1x}-\Psi_{2yyy})=0$$

$$\Psi_2(x,0)=0,\quad \Psi_{2y}(x,0)=0,\quad \Psi_{2y}(x,\infty)=0$$

whose solution is $\Psi_2(x,y)=0$, if we require the solution to be unique. Hence, our first- and second-order solutions coincide with those given by the corresponding usual boundary-layer theory.

Since $\Psi_2(x,y)=0$, the third-order equation is given by

$$\frac{\partial}{\partial y}[\Psi_{3xy}\Psi_{1y}+\Psi_{3y}\Psi_{1xy}-\Psi_{3yy}\Psi_{1x}-\Psi_{3x}\Psi_{1yy}-\Psi_{3yyy}]$$
$$=T[-\Psi_{1y}\Psi_{1xxx}+\Psi_{1x}\Psi_{1xxy}+2\Psi_{1xxyy}]$$
$$+[-6\Psi_{1yy}\Psi_{1xyyy}+12\Psi_{1yyy}\Psi_{1xyy}$$
$$+18\Psi_{1yyyy}\Psi_{1xy}-4\Psi_{1y}\Psi_{1xyyyy}+4\Psi_{1x}\Psi_{1yyyyy}+6\Psi_{1yyyyyy}]$$
$$\equiv g_1+g_2 \qquad (6)$$

where we have defined g_1 and g_2 as the terms in the first and second square brackets, respectively, of the right-hand side.

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‡The last term for $P'^{(2)}_{xy}$ in Ref. 1 should read $(-6T^2/n)/(\partial^3 u/\partial y'^3)$.

We note that, if g_2 were identically zero, then Eq. (6) would be the same as the third-order approximation in the usual boundary-layer theory.[11] However, g_2 is not identically zero. We are interested in the effects of g_2. If we denote the linear differential operator on the left-hand side of Eq. (6) by D then Eq. (6) is just $D(\Psi_3)=g_1+g_2$. If we let $\Psi_3=\Psi_{31}+\Psi_{32}$ and $D(\Psi_{31})=g_1$ then the above equation gives

$$D(\Psi_{32})=g_2 \tag{7}$$

ψ_{31} is well known.[11] We proceed to solve for Ψ_{32} in Eq. (7). Let

$$\Psi_{32}=(1/\sqrt{2x})F(\eta) \text{ with } \eta=y/\sqrt{2x} \tag{8}$$

and keep in mind that $\Psi_1=\sqrt{2x}f(\eta)$ where f is the Blasius function. Thus, Eq. (7) becomes an ordinary differential equation:

$$\frac{d}{d\eta}[F'''+fF''2f'F'-f''F]$$
$$=6f^{(6)}+4ff^{(5)}+12f'f^{(4)}-12\eta f''f^{(4)}-12\eta(f''')^2 \tag{9}$$

where $(\ )'=(d/d\eta)(\ )$ and $f^{(i)}=d^if/d\eta^i$. Using the relation $f'''=-ff''$ repeatedly, together with some standard techniques, one can express the solution of Eq. (9) in the form

$$F(\eta)=(f-\eta f')(f^2-C_3)+C_4f'-2f'\int_0^\eta f(f-\eta f')d\eta$$
$$-(f-\eta f')\int_0^\eta\left[\frac{C_2f'}{f^2f''}-\frac{2f''f'}{f^2}\right]d\eta$$
$$+f'\int_0^\eta\left[\frac{C_2}{f^2f''}-\frac{2f''}{f^2}\right](f-\eta f')d\eta$$
$$-(f-\eta f')\int_0^\eta\frac{f'H}{f^2f''}d\eta+f'\int_0^\eta\frac{H}{f^2f''}(f-\eta f')d\eta \tag{10}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants and

$$H(\eta)=\int_0^\eta(12\eta f''f'''-C_1)fd\eta$$

To insure that the integrands in Eq. (10) remain finite near $\eta=0$ we put $C_2=2\alpha^2$, since for small η, $f(\eta)=\frac{1}{2}\alpha\eta^2+0(\eta^5)$. Before we proceed to determine C_1, C_3, and C_4 we must introduce the following boundary conditions to be associated with Eq. (6):

$$\Psi_{3y}=0 \text{ and } \Psi_{3x}=0 \text{ at } \eta=0$$
$$\omega=-(\Psi_{3xx}+\Psi_{3yy})\to 0 \text{ exponentially as } \eta\to\infty \tag{11}$$

In terms of F the above bondary conditions are satisfied if

$$F(0)=F'(0)=0 \quad F''\to 0 \text{ exponentially as } \eta\to\infty \tag{12}$$
$$\text{and } 3F+5\eta F'\to 0 \text{ exponentially as } \eta\to\infty$$

We see from Eq. (10) that $F(0)=0$ since $f(0)=f'(0)=0$, and $F'(0)=0$ if we choose $C_4=0$. $F''\to 0$ exponentially as $\eta\to\infty$ if we choose

$$C_1=2\frac{\lim_{\eta\to\infty}[(f''+\eta f''')h_1+f'''h^2+\alpha^2/f]}{\lim_{\eta\to\infty}[f''+\eta f''')h_3+f'''h_4+k/f]} \tag{13}$$

where

$$k(\eta)=\int_0^\eta f(\eta)d\eta$$
$$h_1(\eta)=\int_0^\eta\left[\frac{\alpha^2f'}{f^2f''}-\frac{f''f'}{f^2}\right]d\eta$$
$$h_2(\eta)=\int_0^\eta\left[\frac{\alpha^2}{f^2f''}-\frac{f''}{f^2}\right](f-\eta f')d\eta$$
$$h_3(\eta)=\int_0^\eta\frac{kf'}{f^2f''}d\eta$$
$$h_4(\eta)=\int_0^\eta\frac{k}{f^2f''}(f-\eta f')d\eta$$

To satisfy the last condition in Eq. (12) we choose

$$C_3=\beta^2+\frac{1}{3\beta}\lim_{\eta\to\infty}\{[6(f-\eta f')-10\eta f'']h_1-[6f'+10\eta f'']h_2$$
$$+C_1[-3(f-\eta f')+5\eta^2f'']h_3+C_1[3f'+5\eta f'']h_4\} \tag{14}$$

We have made use of the fact that for large $\eta, f\sim\eta-\beta+\gamma\exp[-(\eta-\frac{1}{2}\beta)^2]$ where γ is a constant. Using the relations in Eq. (5), we find from Eqs. (13) and (14) that $C_1=0$ and $C_3=\beta^2$. Furthermore, Eq. (10) can be evaluated to give $F''(0)=\alpha C_3=\alpha\beta^2$. The local coefficient of skin friction up to the third order is given by

$$C_f=\frac{2}{\sqrt{R}}\frac{\partial}{\partial y}(\Psi_1+\frac{1}{\sqrt{R}}\Psi_2+\frac{1}{R}\Psi_{31}+\frac{1}{R}\Psi_{32})y=0 \tag{15}$$

where we made use of the fact that $T=1/R$ for incompressible fluid flow. It is well known[11] that the first three terms in Eq. (15) can be evaluated to give

$$\frac{0.664}{\sqrt{R_x}}+0.551\frac{\log R_x}{R_x^{3/2}}+\frac{C}{R_x^{3/2}}\ (1.2<C<1.5)$$

where $R_x=Rx$ is the local Reynolds number. The appearance of the term in log R_x is derived from the requirement that the vorticity must vanish exponentially. The constant C is undetermined. There is a growing interest in the determination of the value of C (see Note 8 of Ref. 11). The last term in Eq. (15) is new and cannot be predicted by the usual higher-order boundary-layer solutions derived from Navier-Stokes equations. With Ψ_{32} given by Eqs. (8) and (10), we find

$$\frac{2}{\sqrt{R}}\left(\frac{\partial\Psi_{32}}{\partial y}\right)_{y=0}=\frac{1}{R_x^{3/2}}\frac{F''(0)}{\sqrt{2}}=\frac{1}{R_x^{3/2}}\frac{\alpha\beta^2}{\sqrt{2}}=\frac{0.4916}{R_x^{3/2}} \tag{16}$$

Conclusion

The higher-order boundary-layer equations derived from the BKG model of the Boltzman equation are solved by successive approximation. The first- and second-order solutions are found to be identical to the higher-order boundary-layer solutions derived by successively solving the corresponding Navier-Stokes equations. However, the third-order solutions are different. The new higher boundary-layer equations predict a larger coefficient of skin friction, as indicated in Eq. (16).

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Axisymmetric Calculations of Transonic Wind Tunnel Interference in Slotted Test Sections

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Nomenclature

a = slot width
C_p = $(p-p_\infty)/\rho_\infty U_\infty^2/2$, normalized pressure coefficient
$\mathfrak{F}$ = wall b.c. functional, Eq. (1)
F = $r\bar{\varphi}_r$, Eq. (3)
G = $\bar{\varphi} - \ell n(r)\cdot F$, Eq. (3)
l = slot depth
M_∞ = Mach number of reference flow
N = number of slots
p = pressure
p_p = pressure in plenum chamber
p_∞ = pressure in reference flow
q = slot flux/unit length
Q = normalized slot flow potential, Ref. 1
r = radius vector in cross-flow plane
U_∞ = velocity in reference flow
v = normalized slot velocity at y_p, Ref. 1
x = distance along tunnel axis
y_p = coordinate of line in slot center plane where the plenum pressure is imposed, $y_p=0$ at the slot entrance; Ref. 1
γ = specific heat ratio
δ = $(p_p-p_\infty)/\rho_\infty U_\infty^2$; normalized plenum pressure
$\bar{\varphi}$ = perturbation velocity potential to approximate problem, Eq. (2)
$\bar{\varphi}_0$ = integration constant given at beginning of slot, Eq. (6)
ρ_∞ = density of reference flow
FTI = figure of tunnel interference, Eq. (7)

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Introduction

WIND tunnel interference poses a serious problem when testing models in the transonic speed regime. One way to avoid this problem is to use comparatively small models. However, small models will usually yield Reynolds numbers that are too low and large test sections will give too expensive tunnels. A numerical method is urgently required to help resolve these conflicting interests. Recently, a theory for slotted test sections was presented by Berndt.[1] A study has been commenced to investigate numerically the consequences of this theory, and some of the very first results are presented herein. Up to now, only axisymmetric flows have been calculated, although this is no limit to the theory. The wall interference on the model has been defined through a single number, called the figure of tunnel interference (FTI). The FTI is based on an average value of the difference in model surface pressure between the tunnel case and the simulated freestream case. Two different tunnel blocking ratios are demonstrated for a parabolic arc body mounted on a sting at two different Mach numbers, the higher of which gives a fully choked test section. The present calculations only cover test sections with slots of constant width. However, work is now going on with varying slot widths in an attempt to find slot shapes that, hopefully, will give an almost interference-free flow in the test section.

General Outlines of the Theory

The theory of Ref. 1 is built on the calculation of an approximate velocity perturbation potential $\bar{\varphi}$. In comparison with the "exact" solution φ, the approximate $\bar{\varphi}$ is created by averaging (filtering) φ with respect to higher order crossflow variations, caused by the slots and the walls in the test section. By using slender-body cross-flow theory in combination with matched asymptotic expansions, the slot flow is coupled to the averaged potential through a pressure balance equation for each slot. The line y_p along which the plenum pressure δ is specified for each slot is a priori unknown and therefore a part of the total solution. The aforementioned coupling results in a homogeneous wall boundary condition for each slot, giving a relation between $\bar{\varphi}$ and the radial velocity $\bar{\varphi}_r$:

$$\bar{\varphi} = \mathfrak{F}(\bar{\varphi}_r) \tag{1}$$

The functional $\mathfrak{F}$ in Eq. (1) includes the dependence on geometrical data such as the slot width $a(x)$, the depth l, and the number of slots N, as well as the plenum pressure δ. In the general case, a trigonometric interpolation is needed between slots to give the complete outer boundary condition for $\bar{\varphi}$ at the wall surface. The inner boundary condition is the usual slender-body-type approximation of the tangency flow condition, which in the axisymmetric case specifies $r\bar{\varphi}_r$ close to the x axis. At the entrance of the test section, values of $\bar{\varphi}_x$ are coupled to the plenum pressure coefficient, which gives the entrance Mach number. The nonlinear problem for $\bar{\varphi}$ is solved by numerically iterating on the transonic small-perturbation equation between the model and the tunnel wall, repeatedly using relation (1) as an outer wall condition.